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LETTER TO THE EDITOR

The hyperbolic complexification of quantum groups and the isomorphic relations

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Abstract. In this paper we discuss the quantum group $\mathrm{GL}_q(n, H)$ obtained by a hyperbolic complexification of the quantum group $\mathrm{GL}_q(n)$, and prove that $\mathrm{GL}_q(n, H)$ is isomorphic to the direct product $\mathrm{GL}_q(n) \times \mathrm{GL}_q(n)$. In particular, $\mathrm{SU}_q(2, H)$ and $\mathrm{SU}_q(1, 1, H)$ are isomorphic to $\mathrm{SL}_q(2)$.

We have proved [1] that there are the following local isomorphic relations between the hyperbolic complex linear Lie group $\mathrm{GL}(n, H)$ and the real linear Lie group $\mathrm{GL}(n, R)$

$$\begin{array}{ccc} \mathrm{GL}(n, H) & \xrightarrow{\text{local isomorphism}} & \mathrm{GL}(n, R) \times \mathrm{GL}(n, R) \\ \text{embedding} \uparrow & & \uparrow \text{embedding} \\ \mathrm{U}(\eta, H) & \xrightarrow{\text{local isomorphism}} & \mathrm{GL}(n, R) \end{array} \qquad (1)$$

where the n-signature is $\eta = \mathrm{diag}(+1, \ldots, +1, -1, \ldots, -1)$. Therefore a problem is raised: are there the corresponding results for the quantum groups [2–4]? In this letter the answer is given.

In the first place, we discuss what is the hyperbolic complexification of the quantum group $\mathrm{GL}_q(n)$. We use the symbols in the book by Abe [5]. Suppose that A is a commutative ring, and $\mathscr{A}$ is the A-algebra with the unit element $\mathbb{1}$, the structure maps are

$$\eta_{\mathscr{A}}: \quad A \to \mathscr{A} \qquad \mu_{\mathscr{A}}: \quad \mathscr{A} \otimes_A \mathscr{A} \to \mathscr{A} \qquad (2)$$

obeying the relations

$$\mu_{\mathscr{A}}(\mathbb{1} \otimes \mu_{\mathscr{A}}) = \mu_{\mathscr{A}}(\mu_{\mathscr{A}} \otimes \mathbb{1}) \qquad \mu_{\mathscr{A}}(\mathbb{1} \otimes \eta_{\mathscr{A}}) = \mu_{\mathscr{A}}(\eta_{\mathscr{A}} \otimes \mathbb{1}). \qquad (3)$$

Let ε denote the hyperbolic pure imaginary unit, i.e. $\varepsilon^2 = +1$, $\varepsilon \neq \pm 1$. According to common addition and multiplication, the set of all elements in the form of $a + \varepsilon b$ forms a commutative ring H[6], where a and b are real numbers. The characteristic of this ring is that there are mutual nullification divisors [1] z and z^*,

$$z = \frac{1+\varepsilon}{2} \qquad z^* = \frac{1-\varepsilon}{2} \qquad zz^* = 0 \qquad z^2 = z \qquad z^{*2} = z^* \qquad (4)$$

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therefore z and z^* always play the roles 1 or 0, respectively. In the following we only consider the case of $A = R^1$ (the real number ring), and suppose that all elements of $\mathscr{A}$ are independent of ε. We define $\varepsilon x = x\varepsilon$ for $x \in \mathscr{A}$. Let $\mathscr{A}_H$ be the set of all elements of the form $x + \varepsilon y$, $x, y \in \mathscr{A}$. The maps

$$\eta_{\mathscr{A}H}: \quad H \to \mathscr{A}_H \qquad \mu_{\mathscr{A}_H}: \quad \mathscr{A}_H \otimes_H \mathscr{A}_H \to \mathscr{A}_H \tag{5}$$

are defined by

$$\begin{aligned} &\eta_{x+\varepsilon y}(a+\varepsilon b) = \eta_x(a) + \eta_y(b) + \varepsilon[\eta_y(a) + \eta_x(b)] \equiv (a+\varepsilon b)(x+\varepsilon y) \\ &\mu_{\mathscr{A}_H}[(x+\varepsilon y)\otimes(x'+\varepsilon y')] \\ &\qquad = \mu_{\mathscr{A}}(x\otimes x') + \mu_{\mathscr{A}}(y\otimes y') + \varepsilon[\mu_{\mathscr{A}}(x\otimes y') + \mu_{\mathscr{A}}(x'\otimes y)] \\ &\qquad \equiv (x+\varepsilon y)(x'+\varepsilon y'). \end{aligned} \tag{6}$$

It can be easily verified that $\eta_{\mathscr{A}_H}$ and $\mu_{\mathscr{A}_H}$ satisfy (3) while $\mathscr{A}$ is substituted by $\mathscr{A}_H$, therefore $\mathscr{A}_H$ is an H-algebra with the structure maps $\eta_{\mathscr{A}_H}$ and $\mu_{\mathscr{A}_H}$. In this H-algebra we find a simple and interesting identity which plays a key role in our discussions, i.e.

$$\begin{aligned} &[\tfrac{1}{2}(x_1+y_1) + \frac{\varepsilon}{2}(x_1-y_1)][\tfrac{1}{2}(x_2+y_2) + \frac{\varepsilon}{2}(x_2-y_2)] \\ &\qquad = \tfrac{1}{2}(x_1x_2 + y_1y_2) + \frac{\varepsilon}{2}(x_1x_2 - y_1y_2) \qquad x_i, y_i \in \mathscr{A}_H. \end{aligned} \tag{7}$$

Generally, we have

$$\prod_{i=1}^{n} \{\tfrac{1}{2}(x_i+y_i) + \frac{\varepsilon}{2}(x_i - y_i)\} = \tfrac{1}{2}\left(\prod_{i=1}^{n} x_i + \prod_{i=1}^{n} y_i\right) + \frac{\varepsilon}{2}\left(\prod_{i=1}^{n} x_i - \prod_{i=1}^{n} y_i\right) \tag{8}$$

where Π denotes the product in the natural order, i.e. $\Pi_{i=1}^{n} x_i = x_1x_2 \ldots x_n$.

An element of $\mathrm{GL}_q(n)$ can be defined as an $n\times n$ matrix $M = [M_{ij}]$, $M_{ij} \in \mathscr{A}$, and the Yang–Baxter equation is satisfied [4]

$$R(M\otimes\mathbb{1})(\mathbb{1}\otimes M) = (\mathbb{1}\otimes M)(M\otimes\mathbb{1})R \tag{9}$$

where R is the R-matrix. If $\mathscr{A}$ and M, respectively, are substituted by $\mathscr{A}_H$ and $M_H = [M_{(H)ij}]$, $M_{(H)ij} \in \mathscr{A}_H$, then a quantum group $\mathrm{GL}_q(n, H)$ can be similarly defined which is the hyperbolic complexification of $\mathrm{GL}_q(n)$.

We discuss the relation between $\mathrm{GL}_q(n, H)$ and $\mathrm{GL}_q(n)$. Let the map ρ be defined by

$$\begin{aligned} &\rho: \quad \mathrm{GL}_q(n)\times\mathrm{GL}_q(n) \to \mathrm{GL}_q(n, H) \\ &\rho(M, N) = Q \qquad M, N \in \mathrm{GL}_q(N) \\ &Q_{ij} = \tfrac{1}{2}(M_{ij} + N_{ij}) + \frac{\varepsilon}{2}(M_{ij} - N_{ij}). \end{aligned} \tag{10}$$

In the first instance, we must prove that this definition is reasonable, i.e. Q really is a quantum matrix in $\mathrm{GL}_q(n, H)$. The tensor form of the Yang–Baxter equation (9) is

$$R_{ijkl}M_{km}M_{ln} \times M_{jl}M_{ik}R_{klmn}. \tag{11}$$

By using (7) we have

$$R_{ijkl}Q_{km}Q_{ln}=R_{ijkl}\left[\tfrac{1}{2}(M_{km}M_{ln}+N_{km}N_{ln})+\frac{\varepsilon}{2}(M_{km}M_{ln}-N_{km}N_{ln})\right]$$
$$=\left[\tfrac{1}{2}(M_{jl}M_{ik}+N_{jl}N_{ik})+\frac{\varepsilon}{2}(M_{jl}M_{ik}-N_{jl}N_{ik})\right]R_{klmn} \qquad (12)$$
$$=Q_{jl}Q_{ik}R_{klmn}$$

therefore Q is indeed a quantum matrix in $\mathrm{GL}_q(n, H)$.

Let us prove that ρ is an isomorphism. In fact,

$$\rho[(M,N)\cdot(M',N')]=\rho(MM',NN')=P$$
$$P_{ij}=\tfrac{1}{2}(M_{is}M'_{sj}+N_{is}N'_{sj})+\frac{\varepsilon}{2}(M_{is}M'_{sj}-N_{is}N_{sj}). \qquad (13)$$

By using (7) we have

$$P_{ij}=\left[\tfrac{1}{2}(M_{is}+N_{is})+\frac{\varepsilon}{2}(M_{is}-N_{is})\right]\left[\tfrac{1}{2}(M'_{sj}+N'_{sj})+\frac{\varepsilon}{2}(M'_{sj}-N'_{sj})\right]$$
$$=[\rho(M,N)]_{is}[\rho(M',N')]_{sj} \qquad (14)$$

i.e.

$$\rho[(M,N)\cdot(M',N')]=\rho(M,N)\cdot\rho(M',N'). \qquad (15)$$

Next, it is evident that

$$\rho(\mathbb{1},\mathbb{1})=\mathbb{1}. \qquad (16)$$

If $S=[S_{ij}]$ is an arbitrary element of $\mathrm{GL}_q(n, H)$, then S_{ij} can be written as $S_{ij}=x_{ij}+\varepsilon y_{ij}$, x_{ij}, $y_{ij}\in\mathscr{A}$. By the Yang–Baxter equation, we have

$$R_{ijkl}(x_{km}+\varepsilon y_{km})(x_{ln}+\varepsilon y_{ln})=(x_{jl}+\varepsilon y_{jl})(x_{ik}+\varepsilon y_{ik})R_{klmn}. \qquad (17)$$

Since the R-matrix R is independent of the imaginary unit ε, (17) can be split into the real part and the imaginary part

$$R_{ijkl}(x_{km}x_{ln}+y_{km}y_{ln})=(x_{jl}x_{ik}+y_{jl}y_{ik})R_{klmn}$$
$$R_{ijkl}(x_{km}y_{ln}+y_{km}x_{ln})=(x_{jl}y_{ik}+y_{jl}x_{ik})R_{klmn}. \qquad (18)$$

From (18), it is easily seen that if

$$M_{ij}=x_{ij}+y_{ij} \qquad N_{ij}=x_{ij}-y_{ij} \qquad (19)$$

then we have

$$M, N\in\mathrm{GL}_q(n) \qquad \rho(M,N)=S \qquad (20)$$

i.e. the map ρ is full. Therefore the map ρ is indeed an isomorphism,

$$\rho:\quad \mathrm{GL}_q(n)\times\mathrm{GL}_q(n)\approx\mathrm{GL}_q(n,H). \qquad (21)$$

Equation (21) returns to the case of (1) as $q\to 1$; however $\mathrm{GL}(n)$ and $\mathrm{GL}(n, H)$ are not permeated with the limits of $\mathrm{GL}_q(n)$ and $\mathrm{GL}_q(n, H)$, respectively.

The quantum determinant of M is defined by

$$\det_q(M)=\sum_{\sigma\in S_n}^{n}(-q)^{-l(\sigma)}M_{1\sigma_1}M_{2\sigma_2}\ldots M_{n\sigma_n} \qquad (22)$$

where S_n is the symmetric group and $1(\sigma)$ is the length of σ with respect to the simple transformations. The following equation is easily obtained from (8)

$$\det_q[\rho(M, N)] = \frac{1+\varepsilon}{2}\det_q(M) + \frac{1-\varepsilon}{2}\det_q(N) \tag{23}$$

in particular, we have

$$\det_q[\rho(M, N)] = \det_q(M). \tag{24}$$

Therefore we obtain an isomorphism

$$\rho: \quad \mathrm{SL}_q(n) \times \mathrm{SL}_q(n) \approx \mathrm{SL}_q(n, H). \tag{25}$$

Now we discuss the quantum groups $\mathrm{SU}_q(2, H)$ and $\mathrm{SU}_q(1, 1, H)$. Suppose that $\mathscr{A}$ is a C^*-algebra. Let $k = \pm 1$, and signature $\eta(k) = \mathrm{diag}(1, k)$. According to [7, 8], an element of $\mathrm{SU}_q[\eta(k)]$ can be written as a 2×2 matrix g

$$g = \begin{bmatrix} \alpha & -qk\beta^* \\ \beta & \alpha^* \end{bmatrix} \qquad \alpha, \beta \in \mathscr{A} \tag{26}$$

then the Yang–Baxter equation and $\det_q(g) = 1$ are equivalent to

$$\begin{gathered} g\eta(k)g^*\eta^{-1}(k) = \eta(k)g^*\eta^{-1}(k)g = \mathbb{1} \\ g^* = \begin{bmatrix} \alpha^* & \beta^* \\ -qK\beta & \alpha \end{bmatrix}. \end{gathered} \tag{27}$$

Similarly, if an element h of $\mathrm{SL}_q(2)$ is written as [8]

$$h = \begin{bmatrix} a & qb \\ c & d \end{bmatrix} \qquad \tilde{h} = \begin{bmatrix} d & -b \\ -qc & a \end{bmatrix} \qquad a, b, c, d \in \mathscr{A} \tag{28}$$

then the Yang–Baxter equation and $\det_q(h) = 1$ are equivalent to

$$h\tilde{h} = \tilde{h}h = \mathbb{1}. \tag{29}$$

The quantum group $\mathrm{SU}_q[\eta(k), H]$ is the hyperbolic complexification of $\mathrm{SU}_q[\eta(k)]$, $\mathrm{SU}_q[\eta(k), H] \subset \mathrm{GL}_q(2, H)$, and we require that the hyperbolic complex conjugation is included into the $*$-operator extended. Therefore an element g_H of $\mathrm{SU}_q[\eta(k), H]$ can be written as

$$g_H = \begin{bmatrix} x+\varepsilon y & -qk(r^* - \varepsilon s^*) \\ r + \varepsilon s & x^* - \varepsilon y^* \end{bmatrix} \qquad x, y, r, s \in \mathscr{A} \tag{30}$$

and g_H satisfies (27).

Let the map θ be defined as

$$\theta(g) = h_H \equiv \begin{bmatrix} x+y & -qk(r^* - s^*) \\ r+s & x^* - y^* \end{bmatrix}. \tag{31}$$

We have

$$\theta = \pi\rho^{-1} \qquad \theta(g^*) = \eta^{-1}(K)\tilde{h}\eta(K) \tag{32}$$

where ρ is the isomorphism in (25), π is the projective map

$$\begin{gathered} \pi: \quad \mathrm{SL}_q(2) \times \mathrm{SL}_q(2) \to \mathrm{SL}_q(2) \\ \pi(M, N) = M \qquad M, N \in \mathrm{SL}_q(2). \end{gathered} \tag{33}$$

Therefore, from (27) we have

$$\begin{aligned} h_H\tilde{h}_H &= \pi\rho^{-1}(g_H)\eta(K)\pi\rho^{-1}(g_H^*)\eta^{-1}(K) = \pi\rho^{-1}[g_H\eta(K)g^*\eta^{-1}(K)] = \mathbb{1} \\ \tilde{h}_H h_H &= \mathbb{1} \end{aligned} \tag{34}$$

this means $h_H \in \mathrm{SL}_q(2)$. Evidently, θ is an isomorphism

$$\theta: \quad \mathrm{SU}_q[\eta(k), H] \approx \mathrm{SL}_q(2). \tag{35}$$

In the above discussion we obtain the results corresponding to (1) for the case of quantum groups. It can be extended to the general $\mathrm{U}_q(\eta, H)$ and other quantum groups.

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